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ON ABUNDANT NUMBERS

By S. Chowla.

§ 1. We say that the positive integer n is abundant, when

$$\sigma(n) = \sum_{d \mid n} d \geqslant 2n.$$

The probability that a positive integer is abundant is less than $\frac{1}{2}$. More precisely if $A(x)$ is the number of solutions of

$$\sigma(n) \geqslant 2n \qquad (1 \leqslant n \leqslant x)$$

then*

$$0 < \frac{A(x)}{x} < \frac{1}{2}$$

for all $x > x_0$.

In this paper we prove that †

$$(1) \qquad \operatorname*{Lim}_{x \to \infty} \frac{A(x)}{x} = c \ (>0),$$

a result recently conjectured by Felix Behrend. In fact (1)‡ is a simple consequence of a theorem proved by I. Schoenberg in *Mathematische Zeitschrift*, 28, 1928, 171—199.

§ 2. Let

$$(2) \qquad x_{1n},\ x_{2n}, \ldots,\ x_{nn} \qquad (n = 1, 2, 3, \ldots)$$

be n numbers all belonging to the interval $0 \leqslant x \leqslant 1$.

In the paper referred to above Schoenberg has proved that: *Fur die Zahlen* $x_{1n}, x_{2n}, \ldots, x_{nn}$(2) *der strecke* 0,....1 *mögen die Grenzwerte,*

$$(3) \qquad \operatorname*{Lim}_{n \to \infty} \frac{x_{1n}^{k} + x_{2n}^{k} + \ldots\ldots + x_{nn}^{k}}{n} = \mu_k$$

für $k = 1, 2, 3, \ldots\ldots$*vorhanden sein.*

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* Felix Behrend, S.—*B. Preuss. Akad. Wiss*, H. 21/23, 322—328 (1932).

† *Ibid.*, 1933.

‡ (1) has been proved independently by H. Davenport, *ibid.*, 830—837 (1933).

Es existiert eine eindeutig bestimmte für $R(s) > 0$ reguläre und beschränkte Funktion $\Phi(s)$, welche den Bedingungen $\Phi(n) = \mu_n$ fur $n = 1, 2, 3, \ldots$ genugt. Wenn $z(t)$ die Lösung des Momentenproblems,

$$(4) \quad \int_0^1 t^k \, dz(t) = \mu_k \qquad (k = 0, 1, 2, \ldots; \quad \mu_0 = 1)$$

bedeutet, so ist

$$\Phi(s) = \int_0^1 t^s \quad dz(t), \quad R(s) > 0$$

eine Darstellung der Funktion $\Phi(s)$.

Es existiert

$$\lim_{\sigma \to 0} \Phi(\sigma + i\lambda) = \Phi(i\lambda) \qquad (s = \sigma + i\lambda)$$

gleichmassig fur $-\infty < \lambda < \infty$.

Die Zahlen (2) verteilen sich gewiss asymptotisch stetig, wenn

$$(5) \quad \Phi(0) = 1, \text{ und } \lim_{x \to \infty} \frac{1}{x} \int_0^x |\Phi(i\lambda)| \, d\lambda = \theta.$$

Die entsprechende Verteilungsfunktion ist $z(t)$, welches nach den Bedingungen (5) notwendig stetig ist.

§ 3.

Lemma 1: For every complex value of s, we have

$$(6) \quad \lim_{n \to \infty} \frac{\left\{\frac{1^a}{\sigma_a(1)}\right\}^s + \left\{\frac{2^a}{\sigma_a(2)}\right\}^s + \ldots + \left\{\frac{n^a}{\sigma_a(n)}\right\}^s}{n} = \Phi(s)$$

and

$$(7) \quad \Phi(s) = \prod_p \left\{ \left(1 - p^{-1}\right) \left((1 + p^{-1})(1 + p^{-a})^{-s} + p^{-2} (1 + p^{-a} + p^{-2a})^{-s} + \ldots \ldots \right) \right\}$$

where p runs through all primes.

Proof: For $R(s) > 1$ we have

$$\sum_{n=1}^{\infty} \left\{\frac{n^a}{\sigma_a(n)}\right\}^k n^{-s} = \prod_p \left\{ 1 + p^{-s}(1 + p^{-a})^{-k} + p^{-2s}\left(1 + p^{-a} + p^{-2a}\right)^{-k} + \ldots \ldots \right\}$$

$$=\zeta(s).\prod_p\Big\{(1-p^{-s})\Big(1+p^{-s}(1-p^{-a})^{-k}+p^{-2s}(1+p^{-a}+p^{-2a})^{-k}$$
$$+\ldots\ldots\Big)\Big\}$$

$$=\zeta(s).\ F(s)$$

where $\zeta(s)$ is Riemann's zeta function and $F(s)$ is absolutely convergent for $\sigma >$ *Max* $(\frac{1}{2}, 1-a)$. From the latter fact it follows by well-known methods that

$$\lim_{x\to\infty} \frac{\sum_{n\leqslant x}\{n^a/\sigma_a(n)\}^k}{x}=F\ (1).$$

Lemma 2: If $\Phi(s)$ is defined by (7) then

(i) $|\Phi(s)| < 1$ *for* $\sigma > 0$.

(ii) $\Phi(0)=1$.

Proof: From (7),

$$\Phi(0)=\prod_p\Big\{(1+p^{-1})(1+p^{-1}+p^{-2}+\ldots..)\Big\}=1$$

and, for $\sigma>0$,

$$|\ \Phi(s)\ | < \prod_p\Big\{(1-p^{-1})(1+p^{-1}+p^{-2}+\ldots.)\Big\}=1.$$

Lemma 3: If $\Phi(s)$ is defined by (7), then for $a=1$,

(8) $$\lim_{x\to\infty}\frac{1}{x}\int_0^x|\ \Phi\ (i\lambda)|\ d\lambda=0,$$

Proof: We have

$$\Phi(i\lambda)=\prod_p\Big\{(1-p^{-1})\Big(1+p^{-1}(1+p^{-a})^{-i\lambda}$$
$$+p^{-2}(1+p^{-a}+p^{-2a})^{-i\lambda}+\ldots\ldots\Big)\Big\}$$

$$\log|\ \Phi(i\lambda)|=R\sum_p\Big\{\log(1-p^{-1})+\log\Big(1+p^{-1}(1+p^{-a})^{-i\lambda}$$
$$+p^{-2}(1+p^{-a}+p^{-2a})^{-i\lambda}+\ldots.\Big)\Big\}$$

$$=R\sum_p\Big\{-\frac{1}{p}+\frac{(1+p^{-a})^{-i\lambda}}{p}+O(p^{-2})\Big\}$$

$$=-2\sum_p\frac{1}{p}\sin^2\Big\{\frac{\lambda}{2}\log\ (1+p^{-a})\Big\}+O\ (1),$$

$$(9) \qquad |\Phi(i\lambda)| = e^{-2\sum_p \frac{1}{p}\sin^2\left\{\frac{\lambda}{2}\log\left(1+p^{-a}\right)\right\}+O(1)},$$

where the constant implied in O is independent of λ. If we now show that

$$(10) \qquad \lim_{x\to\infty}\frac{1}{x}\int_0^x e^{-2\sum_p \frac{1}{p}\sin^2\left\{\frac{\lambda}{2}\log\left(1+p^{-a}\right)\right\}} d\lambda = 0$$

then (9) shows that (8) is proved.

The proof of (10) is exactly similar to the proof of equation (37) of Schoenberg's paper. Here we have to show that the numbers

$$(11) \qquad \log\left(1+p^{-a}\right)$$

are linearly independent* in Schoenberg's sense. This is easily seen to be true for $a=1$, and lemma 3 is proved. From the *satz* of § 2 and lemmas 1—3 it now follows that

If $0\leqslant\theta\leqslant 1$ *and* $A(x,\theta)$ *is the number of solutions of*

$$\frac{n}{\sigma(n)} \leqslant \theta$$
$$1\leqslant n\leqslant x$$

then

$$\lim_{x\to\infty}\frac{A(x,\theta)}{x}$$

exists and is a continuous function $z(\theta)$ *of* θ.

§ 4. Putting $\theta=1/2$ in the last result we get the result stated in § 1.

3rd September, 1933.

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* It seems likely that the numbers (11) are linearly independent for any $a>o$. The case $a=1$ is proved as in § 19 of Schoenberg's paper.

ON INTEGRAL EQUATION ASSOCIATED WITH PARABOLIC CYLINDER FUNCTIONS

By S. S. Shukrey, M.Sc.,

Research Scholar.

Introduction:

Parabolic cylinder functions have of late been studied by various workers.* The object of the present paper is to study them by the help of the integral equation which these functions satisfy and to study some of their properties. In Art. (1) the integral equation has been obtained, in Art. (2) the recurrence formulæ have been obtained from them. Articles (3) and (4) deal with several expressions of these functions. All the expressions that have been obtained are believed to be new.

I take this opportunity to express my best thanks to Professor S. C. Dhar for his kind help and guidance.

§ 1. Consider the differential equation,

$$(cx^2+b)\frac{d^2y}{dx^2}+cx\frac{dy}{dx}+(bk^2x^2+A)\,y=0 \tag{1}$$

where b, c, A and K are all constants.

We will show that,

$$\phi(x)=\lambda\int\frac{e^{\theta kx}}{(c\theta^2+b)^{\frac{1}{2}}}\,\phi(\theta)\,d\theta$$

satisfies the above equation taken within limits which will be obtained in the course of the analysis. Substituting $\phi(x)$ for y in the above we have

$$\lambda\left\{(cx^2+b)\frac{d^2}{dx^2}+cx\frac{d}{dx}+(bk^2x^2+A)\right\}\int\frac{e^{\theta kx}}{(c\theta^2+b)^{\frac{1}{2}}}\,\phi(\theta)\,d\theta=0$$

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* Lon. Math. Soc. 2, VIII, 1909—10.

Lon. Math. Soc. 2, XVI, 1916—1917.

Jour. of the Ind. Math. Soc., Vol. XIX, No. 8, April, 1932.

Dr. Gorakh Prasad, Proc. Benares Math. Soc., Vol. VIII (1925—28), pp. 47—48.

Whittaker and Watson, Modern Analysis [confluence Hypergeometric functions].

$$\lambda\int\left\{(cx^2+b)k^2\theta^2+cxk\theta+(b^2k^2x^2+A)\right\}\frac{e^{\theta kx}}{(c\theta^2+b)^{\frac{1}{2}}}\phi(\theta)\,d\theta=0$$

$$\lambda\int\left\{(c\theta^2+b)k^2x^2+(c\theta)kx+(bk^2\theta^2+A)\right\}\frac{e^{\theta kx}}{(c\theta^2+b)^{\frac{1}{2}}}\phi(\theta)\,d\theta=0$$

Integrating the left hand side by parts,

$$\lambda\int(c\theta)kx\frac{e^{\theta kx}}{(c\theta^2+b)^{\frac{1}{2}}}\phi(\theta)\,d\theta=\lambda\left[(c\theta^2+b)^{\frac{1}{2}}kx\,e^{\theta kx}\phi(\theta)\right]$$

$$-\lambda\int(c\theta^2+b)^{\frac{1}{2}}kx\left\{kx\,e^{\theta kx}\phi(\theta)+e^{\theta kx}\phi'(\theta)\right\}d\theta.$$

Further integrating by parts the second part of right hand side we have

$$-\lambda\int kx(c\theta^2+b)^{\frac{1}{2}}\left[kx\,\phi(\theta)+\phi'(\theta)\right]e^{\theta kx}\,d\theta=$$

$$\lambda\left\{-e^{\theta kx}\,(c\theta^2+b)^{\frac{1}{2}}\phi'(\theta)\right\}$$

$$+\lambda\int e^{\theta kx}\left[\frac{c\theta}{(c\theta^2+b)^{\frac{1}{2}}}\phi'(\theta)+(c\theta^2+b)^{\frac{1}{2}}\phi''(\theta)\right]d\theta.$$

Hence the total expression will be

$$\lambda\left\{kx(c\theta^2+b)^{\frac{1}{2}}\phi(\theta)\,e^{\theta kx}-(c\theta^2+b)^{\frac{1}{2}}\phi'(\theta)\,e^{\theta kx}\right\}$$

$$+\lambda\int\Big[-k^2x^2(c\theta^2+b)^{\frac{1}{2}}\phi(\theta)\,e^{\theta kx}$$

$$+\frac{kx(c^2\theta^2)\phi'(\theta)+(c\theta^2+b)\phi''(\theta)e^{\theta kx}}{(c\theta^2+b)^{\frac{1}{2}}}\Big]d\theta=0.$$

On simplifying and rearranging we have,

$$\lambda\left[kx\,\phi(\theta)-\phi'(\theta)\right]e^{\theta kx}\,(c\theta^2+b)^{\frac{1}{2}}+$$

$$\lambda\int\left\{(c\theta^2+b)\phi''(\theta)+c\theta\phi'(\theta)+(b\theta^2k^2+A)\phi(\theta)\right\}\frac{e^{\theta kx}}{(c\theta^2+b)^{\frac{1}{2}}}d\theta$$

$$=0.$$

Now ϕ is the same function of θ as it is of x so the expression under the integral sign in above vanishes in virtue of (1). Now in order that the first part may vanish we must have, either $(c\theta^2+b)$ equal to zero, which determines the limits for the integral or $[kx\,\phi(\theta)-\phi'(\theta)]$ equal to zero.

Hence either

$$\phi(x)=\lambda\int_{-\sqrt{bi/c}}^{\sqrt{bi/c}} \frac{e^{\theta kx}}{(c\theta^2+b)^{\frac{1}{2}}}\,\phi(\theta)\,d\theta \tag{2}$$

satisfies equation (1) or

$$\phi(x)=\lambda\int_{k_1}^{k_2} \frac{e^{\theta kx}}{(c\theta^2+b)^{\frac{1}{2}}}\,\phi(\theta)\,d\theta, \tag{3}$$

where k_1 and k_2 are the values of θ for which $\left\{ kx\,\phi(\theta)-\phi'(\theta) \right\}$ vanishes.

Now in equation (1) put $c=\theta$, $b=1$, $A=n+\frac{1}{2}$ and $k=-\frac{1}{4}$ then we have

$$\frac{d^2y}{dn^2}+\left(n+\tfrac{1}{2}-\tfrac{1}{4}x^2\right)y=\theta$$

which is the equation of the parabolic cylinder function (Weber). The integral equation satisfying above will be given by

$$\phi(x)=\lambda\int_{-\infty}^{\infty} e^{\frac{1}{2}i\theta x}\,\phi(\theta)\,d\theta \tag{4}$$

To find out λ we may use Hermite's result

$$D_n(x)=(-1)^n\,e^{\frac{1}{4}x^2}\,\frac{d^n}{dx^n}\left(e^{-\frac{1}{2}x^2}\right)$$

$$D_1(x)=\lambda\int_{-\infty}^{\infty} e^{\frac{1}{2}i\theta x}\,D_1(\theta)\,d\theta$$

$$D_1(x)=\lambda\int_{-\infty}^{\infty} e^{\frac{1}{2}i\theta x-\frac{1}{4}\theta^2}\,\theta\,d\theta$$

$$=\lambda\,e^{-\frac{1}{4}x^2}\int_{-\infty}^{\infty} e^{-\frac{(\theta-xi)^2}{4}}\,\theta\,d\theta$$

$$=\lambda\,2\,xi\,\sqrt{\pi}\;e^{-\frac{1}{4}x^2}$$

Comparing this value from Hermite's result we have $\lambda=\frac{1}{2i\sqrt{\pi}}$; similarly working out the results for $D_2(n)$, $D_3(n)$, $D_4(n)$......

we have λ respectively equal to $\frac{1}{2\sqrt{\pi}(i)^2}$, $\frac{1}{2\sqrt{\pi}(i)^3}$ etc.

Hence

$$D_n(x)=\frac{1}{2\sqrt{\pi}(i)^n}\int_{-\infty}^{\infty} e^{\frac{1}{2}i\theta x}\ D_n(\theta)\ d\theta*$$

where n is an integer.

§ 2. Starting from the above integral equation it is easy to obtain the recurrence formulæ.

$$D_n(x)=\frac{1}{2\sqrt{\pi}(i)^n}\int_{-\infty}^{\infty} e^{\frac{i\theta x}{2}}\ D_n(\theta)\ d\theta$$

$$D_n(x)=\frac{(-1)^n}{2\sqrt{\pi}\ (i)^n}\int_{-\infty}^{\infty} e^{\frac{i\theta x}{2}+\frac{1}{4}\theta^2}\ \frac{d^n}{d\theta^n}\left(e^{-\frac{1}{2}\theta^2}\right)d\theta$$

$$D_{n+1}(x)=\frac{(-1)^{n+1}}{2\sqrt{\pi}(i)^{n+1}}\int_{-\infty}^{\infty} e^{\frac{i\theta x}{2}+\frac{1}{4}\theta^2}\ \frac{d^{n+1}}{d\theta^{n+1}}\left(e^{-\frac{1}{2}\theta^2}\right)d\theta$$

$$=\frac{(-1)^{n+1}}{2\sqrt{\pi}(i)^{n+1}}\int_{-\infty}^{\infty} e^{\frac{i\theta x}{2}+\frac{1}{4}\theta^2}\left\{\frac{d^n}{d\theta^n}\left(e^{-\frac{1}{2}\theta^2}\right)(-\theta)\right\}d\theta$$

$$=\frac{(-1)^{n+1}}{2\sqrt{\pi}(i)^{n+1}}\int_{-\infty}^{\infty} e^{\frac{i\theta x}{2}+\frac{1}{4}\theta^2}\left\{-\theta\frac{d^n}{d\theta^n}\left(e^{-\frac{1}{2}\theta^2}\right)-n\frac{d^{n-1}}{d\theta^{n-1}}\left(e^{-\frac{1}{2}\theta^2}\right)\right\}d\theta$$

$$=\frac{(-1)^{n+2}}{2\sqrt{\pi}(i)^{n+1}}\int_{-\infty}^{\infty} e^{\frac{i\theta x}{2}+\frac{1}{4}\theta^2}\ \theta\frac{d^n}{d\theta^n}\left(e^{-\frac{1}{2}\theta^2}\right)d\theta-$$

$$\frac{(-1)^{n+1}}{2\sqrt{\pi}(i)^{n+1}}\,n\int_{-\infty}^{\infty} e^{\frac{i\theta x}{2}+\frac{1}{4}\theta^2}\left\{\frac{d^{n-1}}{d\theta^{n-1}}\left(e^{-\frac{1}{2}\theta^2}\right)\right\}d\theta$$

$$=\frac{(-1)^{n+2}}{2\sqrt{\pi}(i)^{n+1}}\int_{-\infty}^{\infty} e^{\frac{i\theta x}{2}+\frac{1}{4}\theta^2}\ \frac{d^n}{d\theta^n}\left(e^{-\frac{1}{2}\theta^2}\right)d\theta+x\ D_{n-1}(x) \tag{5}$$

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* I am indebted to Prof. Dhar for this general result of λ.

Also

$$D_n'(x)=\frac{(-1)^n}{2\sqrt{\pi}(i)^n}\int_{-\infty}^{\infty} e^{\frac{i\theta n}{2}+\frac{1}{4}\theta^2}\left\{\frac{d^n}{d\theta^n}\left(e^{-\frac{1}{2}\theta^2}\right)\right\}\frac{i\theta}{2}\,d\theta$$

multiplying the above by -2 and substituting in (6)

we have $D_{n+1}(x)=-2\,D'_n(x)+n\,D_{n-1}(x)$

or $D_{n+1}(x)-n\;D_{n+1}(x)+2\,D'n\,(x)=o,$ (7)

where n is an integer.

But by Watson's* formulae, viz.,

$$D_n(\theta)=e^{-\frac{1}{4}\theta^2}\sum_{m=o}^{\infty}\frac{\Gamma\left(\frac{1}{2}m-\frac{1}{2}n\right)}{m!\;\Gamma-n}(\sqrt{2})^{m-n-2}(-\theta)^m$$

Also

$$D_n(x)=\frac{1}{2\sqrt{\pi}(i)^n}\int_{-\infty}^{\infty} e^{\frac{i\theta x}{2}}\;D_n(\theta)\;d\theta$$

$$D_n(x)=\frac{1}{2\sqrt{\pi}(i)^n}\int_{-\infty}^{\infty} e^{\frac{i\theta x}{2}-\frac{1}{4}\theta^2}\sum_{m=o}^{\infty}\frac{\Gamma\left(\frac{m}{2}-\frac{n}{2}\right)}{m!\;\Gamma-n}(\sqrt{2})^{m-n-2}(-\theta)^m\,d\theta.$$

$$D'_n(x)=\frac{1}{2\sqrt{\pi}(i)^n}\int_{-\infty}^{\infty}\frac{i\theta}{2}\,e^{\frac{i\theta x}{2}-\frac{1}{4}\theta^2}\sum_{m=o}^{\infty}\frac{\Gamma\left(\frac{1}{2}m-\frac{1}{2}n\right)}{m!\;\Gamma-n}(\sqrt{2})^{m-n-2}(-\theta)^m\,d\theta.$$

§ 3.

$$D_n(x)=\lambda\int_{-\infty}^{\infty} e^{\frac{i\theta x}{2}}\;D_n(\theta)\;d\theta$$

$$=\lambda\int_{-\infty}^{\infty} e^{\frac{i\theta x}{2}-\frac{1}{2}\left(\frac{i\theta}{2}\right)^2-\frac{1}{4}x^2}\;e^{-\frac{\theta^2}{8}+\frac{x^2}{4}}\;D_n(\theta)\;d\theta.$$

Also $e^{xt-\frac{1}{2}t^2-\frac{1}{4}x^2}=\sum\limits_{m=o}^{\infty}\frac{D_m(x)}{m!}\,t^m$ †

So $$D_n(x)=\lambda\,e^{\frac{x^2}{4}}\int_{-\infty}^{\infty}\sum_{m=o}^{\infty}\frac{D_m(x)}{m!}\left(\frac{i\,\theta}{2}\right)^m e^{\frac{-\theta^2}{8}}\;D_n(\theta)\;d\theta$$

* Watson Lon. Math. Soc. 2 VIII, 1909—1910.

† Gorakh Prasad, Proc. Benares Math. Soc., Vol. VIII (1925-26), pp. 47, 48.

$$= \lambda e^{x^2/4} \sum_{m=o}^{\infty} \frac{D_m(x)}{m!} \left(\frac{i}{2}\right)^m \int_{-\infty}^{\infty} e^{-\theta^2/8} \theta^m D_n(\theta)\, d\theta^*$$

$$\text{But} \int_{0}^{\infty} e^{(\frac{1}{4}-a)x^2} x^m D_n(x)\, dx = \frac{\sqrt{\pi}\, 2^{\frac{1}{2}n-m-1}\, \Gamma(m+1)}{a^{\frac{1}{2}(m+1)}\, \Gamma(\frac{1}{2}m-\frac{1}{2}n+1)}$$

$$F\left\{-\frac{n}{2}, \frac{m+1}{2}; \frac{m-n}{2}+1, 1-\frac{1}{2a}\right\}$$

putting $a = \frac{3}{8}$ in the above we have

$$\int_{0}^{\infty} e^{-\frac{x^2}{8}} x^m D_n(x) dx = \frac{\sqrt{\pi}\, 2^{\frac{1}{2}n-m-1}\, \Gamma(m+1)}{(\sqrt{3})^{m+1} (\sqrt{2})^{-3(m+1)}\, \Gamma(\frac{1}{2}m-\frac{1}{2}n+1)} \cdot$$

$$F\left\{-\frac{n}{2}, \frac{m+1}{2}; \frac{m-n}{2}+1, -\frac{1}{3}\right\}$$

$$\text{Now} \int_{-\infty}^{\infty} e^{-\frac{\theta^2}{8}} \theta^m D_n(\theta)\, d\theta = \int_{0}^{\infty} e^{-\frac{\theta^2}{8}} \theta^m D_n(\theta)\, d\theta +$$

$$\int_{-\infty}^{0} e^{-\frac{\theta^2}{8}} \theta^m D_n(\theta)\, d\theta$$

$$= \int_{0}^{\infty} e^{-\theta^2/8} \theta^m D_n(\theta)\, d\theta + \int_{0}^{\infty} e^{-\frac{\theta^2}{8}} (-\theta)^m D_n(-\theta) d\theta$$

$$= \left\{1+(-1)^{m+n}\right\} \int_{0}^{\infty} e^{-\frac{\theta^2}{8}} \theta^m D_n(\theta)\, d\theta.$$

$$\therefore D_n(x) = \frac{1}{2\sqrt{\pi}(i)^n} e^{x^2/4} \sum_{m=o}^{\infty} \frac{D_m(x)}{m!} \left(\frac{i}{2}\right)^m \left\{1+(-1)^{m+n}\right\} \cdot$$

$$\frac{\sqrt{\pi}\, 2^{\frac{1}{2}n-m-1}\, \Gamma(m+1)}{(\sqrt{3})^{m+1} (\sqrt{2})^{-3(m+1)}\, \Gamma(\frac{1}{2}m-\frac{1}{2}n+1)} F\left\{-\frac{n}{2}, \frac{m+1}{2};\right.$$

$$\left.\frac{m-n}{2}+1, -\frac{1}{3}\right\}$$

$$D_n(x) = e^{x^2/4} \sum_{m=o}^{\infty} \frac{D_m(x)}{m!}$$

* Watson, Proc. Lond. Math. Soc. (2), VIII, 1910.

$$\frac{(i)^{m-n}\left\{1+(-1)^{m+n}\right\} \Gamma(m+1)}{(\sqrt{3})^{m+1}(\sqrt{2})^{-(n-m-1)} \Gamma(\frac{1}{2}m-\frac{1}{2}n+1)}$$

$$F\left\{-\frac{n}{2}, \frac{m+1}{2} ; \frac{m-n}{2}+1, -\frac{1}{3}\right\} \quad (8)$$

§ 4. We have

$$e^{xt-\frac{1}{2}t^2-\frac{1}{4}x^2} = \sum_{m=o}^{\infty} \frac{D_m(x)}{m!} t^m$$

$$e^{kxt-\frac{1}{2}k^2t^2-\frac{1}{4}x^2} = \sum_{m=o}^{\infty} \frac{D_m(x)}{m!} k^m t^m$$

$$e^{kxt-\frac{1}{2}t^2-\frac{1}{4}x^2k^2} = \sum_{m=o}^{\infty} \frac{D_m(kx)}{m!} t^m$$

$$\therefore e^{-\frac{1}{4}x^2(k^2-1)} e^{-\frac{1}{2}t^2(1-k^2)} \sum_{m=o}^{\infty} \frac{D_m(x)}{m!} k^m t^m = \sum_{m=o}^{\infty} \frac{D_m(kx)}{m!} t^m$$

put $x=r$ and $k=e^{\theta i}$ then

$$e^{-\frac{1}{4}r^2(e^{2\theta i}-1)} e^{-\frac{1}{2}t^2(1-e^{2\theta i})} \sum_{m=o}^{\infty} \frac{D_m(r)}{m!} e^{m\theta i} t^m =$$

$$\sum_{m=o}^{\infty} \frac{D_m(re^{\theta i})}{m!} t^m$$

$$\therefore \sum_{m=o}^{\infty} \frac{D_m(re^{\theta i})}{m!} t^m = e^{-\frac{1}{4}n^2(e^{2\theta i}-1)} \left\{ \sum_{n=o}^{\infty} \frac{\left\{-\frac{1}{2}t^2(1-e^{2\theta i})\right\}^n}{n!} \right\}$$

$$\sum_{m=o}^{\infty} \frac{D_m(r)}{m!} e^{m\theta i} t^m$$

Equating the co-efficient of t^p on either side we have

$$\frac{D_p(re^{\theta i})}{p!} = e^{-\frac{1}{4}r^2(e^{2\theta i}-1)} \sum_{m=o}^{p} \frac{(-\frac{1}{2})^{\frac{p-m}{2}}}{(\frac{p-m}{2})!} (1-e^{2\theta i})^{\frac{p-m}{2}} e^{m\theta i} \frac{D_m(r)}{m!}$$

put $\theta=\frac{\pi}{2}$ and we have

$$\frac{D_p(ri)}{p!} = e^{r^2/2} \sum_{m=o}^{p} \frac{(-\frac{1}{2})^{\frac{p-m}{2}}}{(\frac{p-m}{2})!} 2^{\frac{p-m}{2}} e^{m\pi i} \frac{D_m(r)}{r!}$$

$$= e^{\frac{r^2}{2}} \sum_{m=o}^{p} \frac{(-1)^{\frac{p-m}{2}}}{(\frac{p-m}{2})!} e^{m\pi i} \frac{D_m(r)}{m!}$$

$$=e^{r^2/2}\left\{\frac{(-1)^{\frac{p}{2}}}{(\frac{p}{2})!}\frac{D_0(r)}{1!}+i\frac{(-1)^{\frac{p-1}{2}}}{(\frac{p-1}{2})!}\frac{D_1(r)}{1!}+(-1)\frac{(-1)^{\frac{p-2}{2}}}{(\frac{p-2}{2})!}\cdots\right.$$

$$\left.\frac{D_2(r)}{2!}+i(-1)\frac{(-1)^{\frac{p-3}{2}}}{(\frac{p-3}{2})!}\frac{D_3(r)}{3!}+\cdots\cdots\right\}$$

$$\therefore\frac{Dp(ri)}{p!}=e^{\frac{r^2}{2}}\left\{\frac{(-1)^{\frac{p}{2}}}{(\frac{p}{2})!}\frac{D_0(r)}{1!}+i\frac{(-1)^{\frac{p-1}{2}}}{(\frac{p-1}{2})!}\frac{D_1(r)}{1!}\right.$$

$$+i^2\frac{(-1)^{\frac{p-2}{2}}}{(\frac{p-2}{2})!}\frac{D_2(r)}{2!}+i^3\frac{(-1)^{\frac{p-3}{2}}}{(\frac{p-3}{2})!}\frac{D_3(r)}{3!}+\cdots\cdots$$

$$\left.+(i)^p\frac{Dp(r)}{r!}\right\}.$$

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ON SOME EXPANSIONS AND INTEGRALS INVOLVING THE PARABOLIC CYLINDER FUNCTIONS*

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In Art. 1 of the present paper certain indefinite integrals involving $D_n(z)$ are evaluated, in Art. 2 the expansion of the product $D_o(z)\ D_m(z)\ D_n(z)$ is obtained in series of the functions $D_n(z)$, in Art. 3 the integral $\int_{-\infty}^{\infty} D_o(z)\ D_p(z)\ D_m(z)\ D_n(z)\,dz$ is evaluated and in Art. 4 an Addition theorem is found.

1. *Some indefinite integrals.* $D_n(y)$ is known to satisfy the following formulæ:—

$$\left.\begin{aligned} D'_n(y)+\tfrac{1}{2}y\ D_n(y)-n\ D_{n-1}(y)&=o \\ D_{n+1}(y)-y\ D_n(y)+n\ D_{n+1}(y)&=o \end{aligned}\right\}\dagger \qquad (1\cdot1)$$

$$\left.\begin{aligned} \frac{d}{dy}\left[e^{-\frac{1}{4}y^2}\ D_n(y)\right] &= -e^{-\frac{1}{4}y^2}\ D_{n+1}(y) \\ \frac{d}{dy}\left[y^{n+1}\ e^{-\frac{1}{4}y^2}\ D_n(y)\right] &= -y^n\ e^{-\frac{1}{4}y^2}\ D_{n+2}(y) \end{aligned}\right\}\ddagger \qquad (1\cdot2)$$

From (1·1) we have

$$\frac{d}{dy}\left[e^{-ay^2}\ y^\rho\ D_m(y)\ D_n(y)\right]$$
$$=\left[\rho y^{\rho-1}+(1-2a)\ y^{\rho+1}\right]e^{-ay^2}\ D_m(y)\ D_n(y)$$
$$-e^{-ay^2}\ y^\rho\left[D_m(y)\ D_{n+1}(y)+D_n(y)\ D_{m+1}(y)\right]$$

and

$$\frac{d}{dy}\left[e^{-ay^2}\ y^\rho\ D_{m+1}(y)\ D_{n+1}(y)\right]$$
$$=\left[\rho y^{\rho-1}-(1+2a)\ y^{\rho+1}\right]e^{-ay^2}\ D_{m+1}(y)\ D_{n+1}(y)$$
$$-e^{-ay^2}\ y^\rho\left[(m+1)\ D_m(y)\ D_{n+1}(y)+(n+1)\ D_n(y)\ D_{m+1}(y)\right]$$

*I am indebted to Dr. P. L. Srivastava and Dr. Gorakh Prasad for their kind help and encouragement in the preparation of the paper.

†Whittaker and Watson, Modern Analysis, 4th Edition, p. 350.

‡The formulæ of (1·2) can be deduced from those of (1·1).

It follows therefore that

$$\int e^{-ay^2} y^\rho \left[m\, D_m(y)\, D_{n+1}(y) + n\, D_n(y)\, D_{m+1}(y) \right] dy$$
$$+ \int \left[\rho y^{\rho-1} + (1-2a)\, y^{\rho+1} \right] e^{-ay^2} D_m(y)\, D_n(y)\, dy$$
$$- \int \left[\rho y^{\rho-1} - (1+2a)\, y^{\rho+1} \right] e^{-ay^2} D_{m+1}(y)\, D_{n+1}(y)\, dy$$
$$= e^{-ay^2} y^\rho \left[D_m(y)\, D_n(y) - D_{m+1}(y)\, D_{n+1}(y) \right]^* \quad (1\cdot3)$$

From (1·2) we have

$$\frac{d}{dy} \left[e^{-\frac{1}{2}y^2} y^{m+n+\rho+2} D_m(y)\, D_n(y) \right]$$
$$= y^{m+n+\rho+1} e^{-\frac{1}{2}y^2} \times$$
$$\left[\rho D_m(y)\, D_n(y) - D_m(y)\, D_{n+2}(y) - D_n(y)\, D_{m+2}(y) \right] \quad (1\cdot4)$$

and

$$\frac{d}{dy} \left[e^{-\frac{1}{2}y^2} y^{n+\rho+1} D_m(y)\, D_n(y) \right]$$
$$= y^{n+\rho} e^{-\frac{1}{2}y^2} \times$$
$$\left[\rho D_m(y)\, D_n(y) - D_m(y)\, D_{n+2}(y) - y\, D_n(y)\, D_{m+1}(y) \right] \quad (1\cdot5)$$

whence

$$\int e^{-\frac{1}{2}y^2} y^{m+n+\rho+1} \left[\rho D_m(y)\, D_n(y) - D_m(y)\, D_{n+2}(y) - D_n(y)\, D_{m+2}(y) \right] dy$$
$$= e^{-\frac{1}{2}y^2} y^{m+n+\rho+2} D_m(y)\, D_n(y) \quad (1\cdot6)$$

and

$$\int e^{-\frac{1}{2}y^2} y^{n+\rho} \left[\rho D_m(y)\, D_n(y) - D_m(y)\, D_{n+2}(y) - y\, D_n(y)\, D_{m+1}(y) \right] dy$$
$$= e^{-\frac{1}{2}y^2} y^{n+\rho+1} D_m(y)\, D_n(y) \quad (1\cdot7)$$

As a special case, let us put $\rho = o$ in (1·3), (1·6) and (1·7) then

$$\int e^{-ay^2} \left[m\, D_m(y)\, D_{n+1}(y) + n\, D_n(y)\, D_{m+1}(y) \right] dy$$
$$+ \int y\, e^{-ay^2} \left[(1-2a)\, D_m(y)\, D_n(y) + (1+2a)\, D_{m+1}(y)\, D_{n+1}(y) \right] dy$$
$$= e^{-ay^2} \left[D_m(y)\, D_n(y) - D_{m+1}(y)\, D_{n+1}(y) \right] \quad (1\cdot8)$$

* For the special case $a = o$. *Cf.* R. S. Varma Tohoku Math. Journal, Vol. 34, Part I, 1931.

$$\int e^{-\frac{1}{2}y^2} y^{n+m+1} \left[D_m(y)\, D_{n+2}(y) + D_n(y)\, D_{m+2}(y) \right] dy$$
$$= -e^{-\frac{1}{2}y^2} y^{m+n+2} D_m(y)\, D_n(y) \tag{1·9}$$

and

$$\int e^{-\frac{1}{2}y^2} y^n \left[D_m(y)\, D_{n+2}(y) + y\, D_n(y)\, D_{m+1}(y) \right] dy$$
$$= -e^{-\frac{1}{2}y^2} y^{n+1} D_m(y)\, D_n(y) \tag{1·10}$$

Further if m be equal n we obtain from (1·8), (1·9) and (1·10)

$$2n\int e^{-ay^2} D_n(y)\, D_{n+1}(y)\, dy$$
$$+ \int y\, e^{-ay^2} \left[(1-2a)\, D^2{}_n(y) + (1+2a)\, D^2{}_{n+1}(y) \right] dy$$
$$= e^{-ay^2} \left[D^2{}_n(y) - D^2{}_{n+1}(y) \right] \tag{1·11}$$

$$\int e^{-\frac{1}{2}y^2} y^{2n+1} D_n(y)\, D_{n+2}(y)\, dy$$
$$= -\tfrac{1}{2} e^{-\frac{1}{2}y^2} y^{2n+2} D^2{}_n(y) \tag{1·12}$$

and

$$\int e^{-\frac{1}{2}y^2} y^n \left[D_n(y)\, D_{n+2}(y) + y\, D_n(y)\, D_{n+1}(y) \right] dy$$
$$= -e^{-\frac{1}{2}y^2} y^{n+1} D^2{}_n(y) \tag{1·13}$$

If $a=\frac{1}{2}$ or $-\frac{1}{2}$, (1·11) gives us

$$\int e^{-\frac{1}{2}y^2} \left[2n\, D_n(y)\, D_{n+1}(y) + 2y\, D^2{}_{n+1}(y) \right] dy$$
$$= e^{-\frac{1}{2}y^2} \left[D^2{}_n(y) - D^2{}_{n+1}(y) \right] \tag{1·14}$$

$$\int e^{\frac{1}{2}y^2} \left[2n\, D_n(y)\, D_{n+1}(y) + 2y\, D^2{}_n(y) \right] dy$$
$$= e^{\frac{1}{2}y^2} \left[D^2{}_n(y) - D^2{}_{n+1}(y) \right] \tag{1·15}$$

2. *The expansion of $D_o(z)\, D_m(z)\, D_n(z)$.*

It has been shown that*

$$D_o(z)\, D_{2n}(z) = (-1)^n (2\pi)^{-1/2} e^{\frac{1}{4}z^2} \times$$
$$\sum_{m=o}^{\infty} (-1)^m\, \Gamma\left(\frac{2n+2m+1}{2}\right) \frac{D_{2m}(z)}{\lfloor 2m} \tag{2·1}$$

and

$$D_o(z)\, D_{2n+1}(z) = (-1)^n (2\pi)^{-1/2} e^{\frac{1}{4}z^2} \times$$
$$\sum_{m=o}^{\infty} (-1)^m\, \Gamma\left(\frac{2n+2m+3}{2}\right) \frac{D_{2m+1}(z)}{\lfloor 2m+1} \tag{2·2}$$

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* R. S. Varma, Proc. Benares Math. Soc., Vol. X, pp. 17-18.

Again, if $p \not> n$, then*

$$D_p(z)\ D_n(z)=D_o(z)\ [D_{n+p}(z)+p.nD_{n+p-2}(z)$$
$$+\frac{p.(p-1)}{1\cdot2}\ n(n-1)\ D_{p+n-4}(z)+\dots\ ..] \qquad (2\cdot3)$$

If we substitute the expansions of each of the terms on the right side of (2·3) and collect the co-efficients of similar terms together, we arrive at the following expansions:—

If $p+n$ is even

$$D_o(z)\ D_p(z)\ D_n(z)=(-1)^n\ (2\pi)^{-1/2}\times$$
$$\sum_{m=o}^{\infty}(-1)^m\, a_m\, \frac{D_{2m}(z)}{\lfloor 2m} \qquad (2\cdot4)$$

Where, $a_m=\sum_{r=o}^{p}\frac{(p)!\,(n)!}{r!\,(p-r)!\,(n-r)!}\,\Gamma\left(\frac{n+p+2m-2r+1}{2}\right)$

If $p+n$ is odd

$$D_o(z)\ D_p(z)\ D_n(z)=(-1)^n\ (2\pi)^{-1/2}\times$$
$$\sum_{m=o}^{\infty}(-1)^m\, b^m\, \frac{D_{2m+1}(z)}{\lfloor 2m+1} \qquad (2\cdot5)$$

Where, $b^m=\sum_{r=o}^{p}\frac{(p)!\,(n)!}{r!\,(p-r)!\,(n-r)!}\,\Gamma\left(\frac{n+p+2m-2r+3}{2}\right)$

3. We can now evaluate the definite integral

$$\int_{-\infty}^{\infty}D_o(z)\ D_p(z)\ D_n(z)\ D_m(z)\ dz$$

If $m+n+p$ is odd

$$\int_{-\infty}^{\infty}D_o(z)\ D_p(z)\ D_n(z)\ D_m(z)\ dz=o \qquad (3\cdot1)$$

If $m+n+p$ is even, we have by virtue of the two equations†

$$\int_{-\infty}^{\infty}D_m(z)\ D_n(z)\ dz=o \qquad m\neq n\ (3\cdot2)$$

and $$\int_{-\infty}^{\infty}\{D_n(z)\}^2\,dz=(2\pi)^{1/2}\ n! \qquad (3\cdot3)$$

* Gorakh Prasad, Proc. Benares Math. Soc., Vol. II, p. 18 (1920).
† Whittaker and Watson, *Loc. cit.*, pp. 350-51.

the relation

$$\int_{-\infty}^{\infty} D_o(z)\ D_p(z)\ D_n(z)\ D_m(z)\ dz$$

$$=(-1)^{\frac{m+2n}{2}} \sum_{r=o}^{p} \frac{(p)!\ (n)!}{r!\ (p-r)!\ (n-r)!}\ \Gamma\left(\frac{n+p+m-2r+1}{2}\right)$$

(for the case $p+n$ even)

$$=(-1)^{\frac{2n+m-1}{2}} \sum_{r=o}^{p} \frac{(p)!\ (n)!}{r!\ (p-r)!\ (n-r)!}\ \Gamma\left(\frac{n+p+m-2r+2}{2}\right)$$

(for the case $p+n$ odd) (3·4)

4. *Addition theorem.*

It is known* that

$$e^{ut-\frac{t^2}{2}-\frac{u^2}{4}} = \sum_{o}^{\infty} \frac{D_n(u)}{n!}\ t^n \tag{4·1}$$

From (4·1) we obtain

$$\sum_{o}^{\infty} D_n(u \sin a+v \cos a)\ \frac{t^n}{n!}$$

$$=e^{(u \sin a+v \cos a)\ t-\frac{t^2}{2}-\frac{(u \sin a+v \cos a)^2}{4}}$$

$$=e^{ut \sin a-\frac{t^2 \sin^2 a}{2}-\frac{u^2}{4}} \times$$

$$e^{vt \cos a-\frac{t^2 \cos^2 a}{2}-\frac{v^2}{4}} - e^{-\frac{(u \sin a+v \cos a)^2}{4}+\frac{u^2+v^2}{4}}$$

or

$$\sum_{o}^{\infty} D_n(u \sin a+v \cos a)\ \frac{t^n}{n!}=e^{-\frac{(u \sin a+v \cos a)^2}{4}+\frac{u^2+v^2}{4}} \times$$

$$\left\{\sum_{o}^{\infty} D_n(u)\ \frac{t^n \sin^n a}{n!}\right\}\left\{\sum_{o}^{\infty} D_n(v)\ \frac{t^n \cos^n a}{n!}\right\}$$

Equating the co-efficients of t^n on both the sides we have

$$D_n\ (u \sin a+v \cos a)=e^{\frac{(u \cos a-v \sin a)^2}{4}} \times$$

$$\Big[D_o(u)\ D_n(v) \cos^n a+n\ D_1(u)\ D_{n-1}(v) \sin a \cos^{n-1} a$$

$$+\frac{n(n-1)}{2} D_2(u)\ D_{n-2}(v) \sin^2 a \cos^{n-2} a+\ldots\ldots\ldots\Big]$$

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* Hari Shanker, Proc. Benares Math. Soc., Vol. VI (1924). For a direct proof, *see* Gorakh Prasad, Proc. Benares Math. Soc., Vol. VII—VIII (1925-26).

or

$$D_n(u \sin a + v \cos a) = e^{\frac{(u \cos a - v \sin a)^2}{4}} \times$$

$$\sum_{m=o}^{n} n\, C_m\ D_m(u)\ D_{n-m}(v) \sin^m a \cos^{n-m} a \quad (4{\cdot}2)$$

If we put

$\sin a = \cos a = \frac{1}{\sqrt{2}}$, $u = \sqrt{2}\, x$ and $v = \sqrt{2}\, y$ in (4·2), we have*

$$D_n(x+y) = 2^{-n/2}\, e^{\frac{(x-y)^2}{4}} \times$$

$$\sum_{m=o}^{n} n\, C_m\ D_m(\sqrt{2}\, x)\ D_{n-m}(\sqrt{2}\, y) \quad (4{\cdot}3)$$

* Hari Shanker, Proc. Benares Math. Soc., Vol. VI, p. 14 (1924).

ON THE REDUCIBILITY OF THE GENERAL ELLIPTIC INTEGRAL INTO LOGARITHMS

By K. Venkatachaliengar.

In the following paper I have considered the question of the reduction of the general elliptic integral into logarithms. Abel[1] has considered the problem in an algebraic manner and has given some sufficient conditions. Special cases were considered by Dolbnia[2], Goursat[3] and Halphen[4] and others. My aim in the following paper is to give a set of conditions which do not involve relations with the elliptic transcendentals, namely η, ω, etc. I give in the following paper a set of sufficient conditions for the reducibility, not involving the elliptic transcendentals in any way, which are believed to be new.

For the formation of my conditions the knowledge of the location of the logarithmic critical points is necessary. Consistent with the previous results obtained by others in connection with particular results, these logarithmic critical points are situated at points given by

$$a_r = \frac{2p\omega_1 + 2q\omega_2}{S}$$

where p, q, S are integers and $2\omega_1$, and $2\omega_2$ are the periods of the elliptic integral. Although this condition appears to be involving the transcendentals it is known that the condition can be given in a rationalized form. After this condition, I have given two other conditions, one of them being necessary, *viz.*, the co-efficient of the integral function corresponding to the sum of all the integrals of the second kind. should be zero. The other condition is towards creating the sufficiency. The last two conditions, after the knowledge of the location of the logarithmic critical points, are formed with the co-efficients of the function that is to be integrated, and radicals involving the logarithmic critical points. This is, however, the general case. If some more conditions which are given in the body of the paper, as well as the conditions set forth below are satisfied, then the integral reduces itself into logarithms.

1. Oevres.
2. Oevres.
3. Series of papers published in Bull. de la Soc. Maths. (1880-1900).
4. Functions Elliptiques.

~~The conditions are—~~

Let the logarithmic critical points be given by

$$\left\{ a_1, a_2, a_3, \ldots\ldots\ldots a_n \right\}.$$

Then the set of conditions are

$$(1) \ldots \begin{cases} l_1 a_1 + l_2 a_2 + \ldots\ldots\ldots\ldots + l_r a_r = \dfrac{2p_1 \omega_1 + 2q_1 \omega_2}{S_1} \\ l_{r+1} a_{r+1} + \ldots\ldots\ldots\ldots + l_t a_t = \dfrac{2p_2 \omega_1 + 2q_2 \omega_2}{S_2} \\ l_{t+1} a_{t+1} + \ldots\ldots\ldots\ldots + l_x a_x = \dfrac{2p_3 \omega_1 + 2q_3 \omega_2}{S_3} \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ l_s a_s \ldots\ldots\ldots\ldots\ldots\ldots + l_n a_n = \dfrac{2p_k \omega_1 + 2q_k \omega_2}{S_k} \end{cases}$$

where all the letters except the a's are integers, positive or negative.

Suppose we consider the integral of the form

$$(2) \ldots \int \frac{a_m x^m + a_{m-1} x^{m-1} + \ldots + a_o}{b_n x^m + \ldots\ldots\ldots\ldots + b_o} \cdot \frac{dx}{\sqrt{4x^3 - g_2 x - g_3}}.$$

Then dividing the numerator by the denominator (if $m>n$) and substituting $x=p\ (u,\ g_2,\ g_3)$ we obtain two integrals

$$(3) \ldots\ldots \int (k_o + k_1 p + \ldots\ldots\ldots\ldots + k_q p^q)\, du$$

$$(4) \ldots + \int \left(\frac{c_o + c_1 p + c_2 p^2 + \ldots\ldots\ldots + c_s p^s}{(p-a_1)^{\lambda_1} (p-a_2)^{\lambda_2} \ldots\ldots (p-a_r)^{\lambda_r}} \right) du$$

where $a_r = p\ (a_r)$, $(r=1, 2, \ldots\ldots\ldots n)$.

First of all, it is easily seen that the condition for the reducibility of the integral into logarithms is that the integral should consist of a number of logarithms of elliptic functions and terms expressed rationally in terms of the elliptic functions used, *i.e.*,

$$p(u) \text{ and } p'(u)$$

Hence in order to find a criterion for the reducibility it is enough to consider the terms that are not periodic. Such terms will be of the following three forms:—

$$\text{(a) } u, \quad \text{(b) } \zeta(u), \quad \text{(c) } \operatorname{Log}\left\{ \sigma(u+a_r) \right\}.$$

Now first of all $\zeta(u)+du$, where d is a constant can never represent a doubly periodic function; for, if it does, then

$$2\,d\omega_1 - \eta_1 = 0$$
$$\text{and} \quad 2\,d\omega_2 - \eta_2 = 0;$$

This is certainly inconsistent with the relation

$$\eta_1\,\omega_2 - \eta_2\,\omega_1 = \pi i$$

and $\zeta(u)$ combined with terms in (a) and in (c) can never be represented as a number of logarithms of elliptic functions, for, $e^{\zeta(u)}$ has an isolated essential singularity at $u=0$ which can never be made regular by the addition of any number of terms of the form (a) and (c). Hence for the reducibility it is necessary that the co-efficient of $\zeta(u)$ should vanish. This co-efficient is evidently rationally expressible in terms of the a_r's and the co-efficients of the various terms in the original integral (2), and

$$\sqrt{4a^3{}_r - g_2\,a_r - g_3}\text{'s.}$$

On returning to our original integral (2) I determine the co-efficients of $\zeta(u)$ and u. We know that the polynomial in integral (3) can be written in the form

$$d_0 + d_1\,p(u) + d_2\,p'(u) + \ldots\ldots + d_k\,p^{(k)}(u);$$

hence, we see that the co-efficients of u and $\zeta(u)$ in integral (3) are d_0 and d_1 respectively. Next as regards (4), on expressing the fraction in the integral in partial fractions, we arrive at expressions of the form

$$\int \frac{du}{[p(u)-p(a)]^r};$$

now we can calculate this integral by means of known recurrence formulae. As the form of one term in it is wanted for my conditions I propose to obtain it in this way.

$$\frac{p'(a)}{p(u)-p(a)} = \zeta(u+a) - \zeta(u-a) - 2\zeta(a).$$

Differentiating this successively with respect to a, we get

$$\frac{1}{[p(u)-p(a)]^r} \doteq f[p(a),\,p'(a)] \times \mathrm{M} + \ldots\ldots\ldots\ldots$$
$$\ldots\ldots + \Lambda\,[p(a),\,p'(a)] \left\{ p(u+a) + p(u-a) \right\}$$
$$+ \Psi\,[p(a),\,p'(a)]$$

Where

$$\mathrm{M} = \zeta(u+a) - \zeta(u-a) - 2\zeta(a);$$

and the functions f, Λ and Ψ are rational functions of $p(a)$ and $p'(a)$.

And finally as $\zeta(u+a)-\zeta(u)$ is an elliptic function we can write the expression after integrating the integral (4), in the following form:—

$$\sum_a f_a \log e^{-2\zeta(a)\,u} \frac{\sigma(u+a)}{\sigma(u-a)} + e_0\, u + c_1\, \zeta(u).$$

+ terms which can be expressed rationally in terms of $p(u)$, and $p'(u)$. Hence our integral (2) is pseudo-elliptic if the following function, *viz.*,

$$\sum_a f_a \log e^{-2\zeta(a)\,u} \frac{\sigma(u+a)}{\sigma(u-a)} + (e_o+d_o)\, u + (e_1+d_1)\, \zeta(u)$$

is expressible as a finite combination of logarithms of elliptic functions.

Now I propose to find a set of sufficient conditions for the reducibility.

The first necessary condition that is obtained is, of course,

$$(A) \ldots\ldots\ldots\ldots (e_1+d_1)=0.$$

Now suppose we write the other terms in the following form

$$k\, u + \sum_a f_a \log e^{[-2\zeta(a)-\lambda_a]u} \times \frac{\sigma(u+a)}{\sigma(u-a)}$$

where the λ_a's are arbitrary which are fixed later on, and k is expressible in terms of λ's, a_r's and $\sqrt{4\, a_r^3 - g_2\, a_r - g_3}$'s rationally.

So also is the condition $e_1+d_1=0$.

Now we shall find the condition for the expression

$$f_a \log e^{-[2\zeta(a)+\lambda_a]u} \times \frac{\sigma(u+a)}{\sigma(u-a)}$$

to be the logarithm of an elliptic function [of course, I always mean, with the same periods] writing this in the form

$$\frac{f_a}{R_a} \log e^{-R_a[2\zeta(a)+\lambda_a]u} \times \left[\frac{\sigma(u+a)}{\sigma(u-a)}\right]^{R_a}$$

where R_a is an integer.

The condition for

$$e^{-R_a[2\zeta(a)+\lambda_a]u} \times \left[\frac{\sigma(u+a)}{\sigma(u-a)}\right]^{R_a}$$

to be an elliptic function with $2\omega_1$ and $2\,\omega_2$ as periods is obtained easily, *viz.*,

$$-\omega_1\, [2\zeta(a)+\lambda_a]+\eta_1\, a=\frac{r\,\pi\, i}{R_a}$$

$$-\omega_2\,[2\zeta(a)+\lambda_a]+\eta_2\,a=\frac{s\,\pi\,i}{R_a}$$

where r and s are integers. Using the fact that

$\eta_1\omega_2-\eta_2\omega_1=\pi i$, we obtain the condition for this to be

$$a=\frac{2\,r\omega_1+2\,s\omega_2}{R_a}$$

$$\zeta\,(a)+\frac{\lambda_a}{2}=\frac{2\,r\eta_1+2\,s\eta_2}{R_a}$$

and corresponding conditions for the remaining terms. Here the second condition by means of an artifice can be put in a rational form in the following way.

The condition is

$$\frac{R_a\,\lambda_a}{2}=r\eta_1+s\eta_2-R_a\,\zeta\left[\frac{2\,r\omega_1+2\,s\omega_2}{R_a}\right]$$

Now we know that

$Z(a)=\zeta[R_a-1)a]-(R_a-1)\,\zeta(a)$ can be expressed rationally in terms of $p(a)$ and $p'(a)$. Now this function is evidently equal to

$$-\zeta\,(a)-(R_a-1)\;\zeta\,(a)+r\eta_1+s\eta_2$$

$$=-R_a\,\zeta\,(a)+r\eta_1+s\eta_2=\frac{R_a\lambda_a}{2}$$

Hence the λ's are determined rationally in terms of $[p(a_r)]$'s and $[p'(a_r]$'s; and therefore k is also expressible in terms of the above functions rationally.

The other condition for reducibility after the condition that are indicated for the a's are satisfied is that k, which depends upon the λ's should be equal to zero. Hence this is the other condition which is certainly rational in

(a_r)'s and $[\sqrt{4\,a_r^3-g_2\,a_r\,-\,g_3}\,]$'s.

In order to obtain a rational condition in place of the condition that are to be satisfied in the case of the a's. We may proceed in the following way. Now from the expression of $(p(a), p(2a), p(3a), \ldots$ which are formed rationally in terms of

a_r, and $\sqrt{4\,a_r^3-g_2\,a_r-g_3}$;

If this series is such [and it is obvious that if we have $a=\dfrac{2r\omega_1+2s\omega_2}{R_a}$, then $p[R_a-1)a]=p(a)]$ that after a certain

stage we get one term in the series which is equal to $p(a)$, and the next term becomes infinite, and the other terms once again become equal to $p(a)$, $p(2a)$, . . . then $a = \frac{2r\omega_1 + 2s\omega_2}{R_a}$. Hence the conditions that the a's should satisfy can be put in the above form.

Now this is a general condition, but in particular cases it may so happen that the criterion is widened. For illustration, I have taken the case where there are only five logarithmic critical points, *viz.*, a, β, γ, δ, ϵ. But the general case can also be dealt with similarly.

Suppose that integers l, m, n, P, Q exist such that

$$\frac{f_a}{l} = \frac{f_\beta}{m} = \frac{f_\gamma}{n} \text{ and } \frac{f_\delta}{P} = \frac{f_\epsilon}{Q}$$

Then we can combine the terms in the following way

$$\frac{f_a}{lR_a} \log \left\{ e^{-[2l\zeta(a) + 2m\zeta(\beta) + 2n\zeta(\gamma) + \lambda_a]R_a u} \times \left[\frac{\sigma(u+a)}{\sigma(u-a)}\right]^{lR_a} \times \left[\frac{\sigma(u+\beta)}{\sigma(u-\beta)}\right]^{lR_\beta} \times \left[\frac{\sigma(u+\gamma)}{\sigma(u-\gamma)}\right]^{lR_\gamma} \right\}$$

$$+ \frac{f_\delta}{PR_\delta} \log \left\{ e^{-[2P\zeta(\delta) + 2Q\zeta(\epsilon) + \lambda_\delta]R_\delta u} \times \left[\frac{\sigma(u+\delta)}{\sigma(u-\delta)}\right]^{PR_\delta} \times \left[\frac{\sigma(u+\epsilon)}{\sigma(u-\epsilon)}\right]^{QR_\epsilon} \right\}$$

Now the conditions in this case are

$$la + m\beta + n\gamma = \frac{2r\omega_1 + 2s\omega_2}{R_a}$$

$$\text{and} \quad P\delta + Q\epsilon = \frac{2p\omega_1 + 2q\omega_2}{R_\delta};$$

which again can be put in the proper rational form in a similar manner

and two other conditions which are

$$\lambda_a = -l\zeta(a) - m\zeta(\beta) - n\zeta(\gamma) + \frac{r\eta_1 + s\eta_2}{R_a}$$

$$\lambda_\delta = -P\zeta(\delta) - Q\zeta(\epsilon) + \frac{p\eta_1 + q\eta_2}{R_\delta}$$

Here also by means of a similar artifice we can express the λ's rationally in terms of

(a_r)'s and $[\sqrt{4a_r^3 - g_2 a_r - g_3}]$'s.

Now

$\zeta(l\alpha+m\beta+n\gamma)-l\zeta(\alpha)-m\zeta(\beta)-n\zeta(\gamma)$ can be expressed rationally in terms of $p(\alpha)$, $p'(\alpha)$, $p(\beta)$, $p'(\beta)$, etc. Hence λ_α can be expressed in terms of the above quantities, and so is λ_δ expressible. Therefore the condition corresponding to $k=0$ is also expressed rationally in terms of

$$(a_r)\text{'s and } [\sqrt{4\,a_r^3-g_2\,a_r-g_3}\,]\text{'s.}$$

As a corollary the following interesting results can be obtained *viz.*:

If a general elliptic integral which consists only of integrals of the first and ~~second~~ third kinds [in that case only it is reducible to logarithms] is such that the places at which the integral has logarithmic critical points, namely $[a_1, a_2, a_3, \ldots a_n]$ are such that

$$a_r=\frac{2\,p_r\omega_1+2\,q_r\omega^2}{k_r}\,,\quad (r=1, 2, 3, \ldots n,)$$ then by adding an integral of the form

$$k\int\frac{d_x}{\sqrt{4\,x_3-g_2\,x-g_3}}$$ it is reducible into logarithms.

HEILBRONN'S CLASS-NUMBER THEOREM

By S. Chowla.

This note contains a slightly modified version of Heilbronn's proof of his class-number theorem.[1] In particular my proof is independent of the theory of ideals.

1. The notation used here is the same as in Heilbronn's paper ("On the class-number in imaginary quadratic fields," to be published shortly). The following additional ideas are introduced.

Let $\chi_1(n)$ denote a real primitive character $(mod\ m_1)$ $(m_1 > o)$ such that

$$\sum_{n=1}^{\infty} \chi_1(n)\, n^{-s} = O$$

for some $s=\rho$ in the half-plane $\sigma > \frac{1}{2}$.

Let $p_1, p_2, p_3 \ldots\ldots$ denote the primes in ascending order of magnitude which are not contained in m_1. We now choose m so that

$$(1) \quad m = m_1\, p_1\, p_2\, p_3 \ldots\ldots p_r$$

where r will be defined later.

We define $\chi(n)$ a character $(mod\ m)$ so that

$$\chi(n) = \chi_1(n) \qquad [(n, m) = 1],$$

$$\chi(n) = O \qquad [(n, m) > 1].$$

It follows that

$$L_o(s) = L(s, \chi) = \sum_1^{\infty} \chi(n)\, n^{-s}$$

vanishes for $s=\rho$ where

$$(2) \quad \rho = \theta + i\,\phi \qquad (\tfrac{1}{2} < \theta < 1).$$

A is an absolute positive constant[2] such that for $1 \leqslant l_2 \leqslant m$, $\frac{1}{2} < \sigma < 1$, we have

$$(3) \quad m^{-2\sigma}\ \zeta\left(2s-1, \frac{l_2}{m}\right) = O\ (mA-2), \quad [A>2].$$

---

(1) Equation (11) of this paper. $H=h\ (d)$ is the number of primitive classes of binary quadratic forms of negative discriminant $-d$.

(2) It is easy to prove the existence of such an A.

We define r as the least positive integer satisfying

$$(4) \quad |d|^{\frac{\theta}{4}-\frac{1}{6}} < m^A < |d|^{\frac{\theta}{2}-\frac{1}{3}}$$

where[3] $\theta > \frac{3}{4}$. Since m_1 is fixed it follows from (1) and (4) that

$$(5) \quad r \to \infty \text{ as } -d \to \infty .$$

The constants implied in our O symbols depend only on m_1, s and ρ but they are independent of r, d and H.

2. *Proof of the theorem.*—Following Heilbronn we prove that (for details see the concluding section 3).

$$(6) \quad 0 = Z(2\rho) \prod_{p/m} (1-p^{-2\rho}) \sum_a \chi(a)\, a^{-\rho} + O\left(Hm^A\, |d|^{\frac{1}{4}-\frac{\theta}{2}}\right) + O\left(Hm^A\, |d|^{-\frac{1}{2}\theta}\right).$$

Further, since $\chi(a)$ is O unless a is prime to m, it follows that

$$(7) \quad \left|\sum_a \chi(a)\, a^{-\rho}\right| \geqslant 1 + O\left(\sum_{\substack{a \\ a>k}} a^{-\rho}\right)$$

where k is the least positive integer prime to m. From (1) and (5) we obtain:

$$(8) \qquad k \to \infty \text{ as } -d \to \infty .$$

Since

$$(9) \qquad \sum_{\substack{a \\ a>k}} a^{-\rho} = O\left(Hk^{-\theta}\right)$$

it follows from (9), (8) and (7) that if H is bounded, then

$$(10) \qquad \left|\sum_a \chi(a) a^{-\rho}\right| \geqslant \tfrac{1}{2}.$$

If H is bounded the equations (4), (6) and (10) contain a contradiction as $-d \to \infty$, since the moduli of the terms

$$Z(2\rho) \text{ and } \prod_{p/m} (1-p^{-2\rho})$$

are greater than absolute constants ($\theta > \frac{3}{4}$).

Hence

$$(11) \qquad H = h(d) \to \infty \text{ as } -d \to \infty ,$$

the desired result.

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(3) If $\theta < \frac{3}{4}$ for all m_1, then by a theorem of Hecke, $H = h(d) \to \infty$ as $-d \to \infty$.

3. From (3) and the proof of lemma 9 of Heilbronn's paper it follows that[4]

$$(12)\quad \psi(s)=O\left(mA-2\,|\,d\,|^{\frac{1}{4}-\frac{\sigma}{2}}+|\,d\,|^{-\frac{1}{2}\sigma}\right).$$

From (12) and the proof of lemma 10 of Heilbronn's paper it follows that[5]

$$(13)\quad L_o(s)\ L_2(s)=Z(2s)\prod_{p/m}(1-p^{-2s})\ \sum_{a}\chi(a)\,a^{-s}$$

$$+O(HmA\,|\,d\,|^{\frac{1}{4}-\frac{1}{2}\sigma}+Hm2\,|\,d\,|^{-\frac{1}{2}\sigma}),$$

whence putting $s=\rho$ we get

$$(14)\quad O=Z(2\rho)\prod_{p/m}(1-p^{-2\rho})\ \sum_{a}\chi(a)\,a^{-\rho}$$

$$+O(HmA\,|\,d\,|^{\frac{1}{4}-\frac{1}{2}\theta}+Hm2\,|\,d\,|^{-\frac{1}{2}\theta})$$

and this is the same as (6) since $A>2$.

---

(4) $\psi(s)=\sum_{\substack{y=1\\ y\equiv l_2(m)}}^{\infty}\ \sum_{\substack{x=-\infty\\ x\equiv l_1(m)}}^{\infty}(ax^2+bxy+cy^2)^{-s}$ for $\sigma>1$.

(5) For $\sigma>\frac{1}{2}$, $s\neq 1$.

COLLINEATIONS IN PATH-SPACE

By D. D. Kosambi (*Poona, India*).

A geometry attached to systems of second order differential equations of the generic type

$$(1) \quad \ddot{x}^i + a^i(x, \dot{x}, t) = 0 \quad \dot{x}^i = \frac{dx^i}{dt} \text{ etc.} \quad (i = 1 \ldots n)$$

has been discussed elsewhere[1]. Curves representing solutions of (1) can be regarded as the generalized autoparallel lines or *paths* of a space, and the intrinsic differential geometry thereof is developed from two main assumptions: (*a*) the tensor invariance of all fundamental equations, including (1), and (*b*) the existence of a vectorial operator, the vanishing of which defines a parallelism making solutions of (1) autoparallel lines.

I here attempt to investigate a special type of path-space which allows continuous groups of deformations carrying paths into paths.

Let $u^i(x)$ be a vector field representing an infinitesimal transformation of such a group by means of the "small displacement"

$$\bar{x}^i = x + u^i \, \delta\xi$$

Then the functions u^i must satisfy the equations of variation of (1):

$$(2) \quad \ddot{u}^i + a^i_{;r} \dot{u}^r + a^i_{,r} u^r = 0$$

As usual, a repeated index denotes summation; moreover, $f_{;k} = \frac{df}{d\dot{x}^k}$ and $f_{,k} = \frac{df}{dx^k}$. Inasmuch as the operator for total differentiation with respect to t is

$$\frac{d}{dt} = -a^r \frac{d}{d\dot{x}^r} + \dot{x}^r \frac{d}{dx^r} + \frac{d}{dt}$$

we find that (2) reduce to

$$(2') \quad u^i_{,m,r} \dot{x}^m \dot{x}^r - a^r u^i_{,r} + a^i_{;j} u^j_{,r} \dot{x}^r + a^i_{,j} u^j = 0.$$

Let it be further assumed that a^i has the form of a polynomial in $\dot{x}$:

$$(3) \quad a^i = A^i + A^i_h \dot{x}^h + \Gamma^i_{hr} \dot{x}^h \dot{x}^r + \ldots + A^i_{h_1 \ldots h_m} \dot{x}^{h_1} \ldots \dot{x}^{h_m} + \ldots$$

---

(1) D. D. Kosambi *Rendiconti della Reale Accademia dei Linceit* pp.

D. D. Kosambi *Math. Zeitschrift, Bd.* 37, (1933), pp. 608, 618.

The coefficients A, Γ are functions of x alone, symmetric in all subscripts; the letter Γ has been used for the quadratic terms only, for reasons that will be apparent later.

In the previous papers referred to, as well as in a remarkable exposition by M. Cartan[2], it was shown that

$$a^i - \tfrac{1}{2}\dot{x}^r a^i{}_{;r} \quad \text{as also} \quad a^i{}_{;l;m;n}$$

and their further partial derivatives with respect to $\dot{x}$ are tensors of the rank expressed by the indices. It follows, since (1) are tensor invariant, that:

In a polynomial a^i of the form (3), the terms of any degree except two have tensor co-efficients ($A^i \ldots$). The coefficients of the second degree terms ($\Gamma^i{}_{jk}$) have the same laws of transformation as those of a symmetric affine connection.

We can, therefore, obtain a covariant differentiation with respect to the Γ's alone, by the usual rules:

$$(4) \qquad \lambda^i{}_{|h} = \lambda_{,h} + \lambda^r \, \Gamma^i{}_{hr}$$

and so on for tensors of any rank.

The equations (2′) also represent polynomials in $\dot{x}$, which must vanish identically, as our infinitesimal transformations form vector fields independent of the particular paths chosen. We thus obtain, from terms not of the second degree in $\dot{x}$,

$$(5) \quad u^m{}_{|r}\big[A^i{}_{mh_2..h_j}\,\delta^r{}_{h_1} + A^i{}_{h_1m..h_j}\,\delta^r{}_{h_2} + \quad + A^i{}_{h_1h_2..m}\,\delta^r{}_{h_j} - \delta_{im}\,A^r{}_{h_1h_2..h_j}\big] + u^m A^i{}_{h_1h_2..h_j/m} = o.$$

where the vertical bar before a subscript denotes covariant differentiation with respect to $\Gamma^i{}_{jk}$ as defined in (4); $\delta^i{}_j$ are the usual Kronecker symbols, zero or unity in value as the two indices are different or coincident. The second degree terms, however, give:

$$(6') \quad u^i{}_{|j|n} + u^i{}_{|k|j} = u^l\big[R^i{}_{jki} + R^i{}_{kjl}\big].$$

$R^i{}_{jkl}$ being the curvature tensor for the Γ's. But with the following identities:—

$$R^i{}_{jkl} + R^i{}_{jlk} = 0$$

$$(7) \qquad R^i{}_{jkl} + K^i{}_{klj} + R^i{}_{ljk} = 0$$

$$u^i{}_{|j|k} - u^i{}_{|k|j} = -R^i{}_{hjk}\,u^h$$

we can reduce this to the normal form

$$(6) \qquad u^i{}_{|j|k} = R^i{}_{jkl}\,u^l\,.$$

(2) E. Cartan *Math. Zeitschrift, Ibid.*, pp. 619, 622.

We have thus broken up the equations of variation into one system of partial differential equations of the second order, and several of the first order, all being tensorial in form.

The problem of determining whether any solutions of (5) and (6) exist is reducible to one of algebra[3], though not explicitly soluble as a rule. The general solution, if any exist, can be expressed in terms of p independent fundamental solutions ($p \leqq n^2+n$) as a linear combination of these with constant coefficients. But (6) has a further very important property, easily proved by means of its compatibility conditions and the identities (7). That is, if u^i, v^i be any two distinct solutions, the alternant or Poisson bracket

$$(u, v)^i \equiv u^r v^i_{,r} - v^r u^i_{,r} \equiv u^r v^i{}_{|r} - v^r u^i{}_{|r}$$

is also a solution of (6). Thus our independent infinitesimal transformations generate a group. It does not by any means follow that the common solutions of [5] and [6] generate a Lie group. This is the case, however, when none or only one such common solution exists, apart from this trivial case, the most general conditions can again be reduced to a problem of algebra, and in fact to the discussion of the independence of a series of linear or bilinear forms. One might consider the possibility of [5] being a consequence of the compatibility conditions of [6], or, of the equations [5] themselves possessing the group property. In general, it would not seem that such multiparameter groups exist when the a^i contain terms of degree higher than two in $\dot{x}$. The main point is that there exists a covariant derivation as for the affine connections, and that the general operation of the *derivate* or *biderivate* which I have elsewhere defined, can be replaced by a known and familiar type. The analytic connections are not more general than those of the form $a^i = \Gamma^i_{jk}\, \dot{x}^j \dot{x}^k + E^i_j\, \dot{x}^j + \omega^i$ which, by the way, are the only ones that are analytic in the space of $n+1$ dimensions wherein t is taken as one of the x's.

The same discussion for the most general form of a^i has no meaning, but is easily extensible to a^i that are analytic in $\dot{x}$ and sufficiently differentiable in x to allow a discussion of compatibility conditions. Even more, convergence of the infinite series can be ignored if merely an expansion of the prescribed form exists. Formally, each power of $\dot{x}$ in the expansion yields just one equation, independent of all other terms except those of the second degree. Apart from the question of solving an infinite set of differential equations (present also in the analytic case) the only difficulty possible would be that of the absence of uniqueness of

(3) L. P. Eisenhart *Non-Riemannian Geometry* (1927), pp. 126, 132.

expansion. But in this last case, if it can occur at all, we may regard the various forms as given by the use of different ways of describing the same space; or as different spaces that are feasible for the same paths. Similarly, asymmetric components, corresponding to the torsion tensor and the like can be introduced in the various coefficients, though they will not appear in the actual equations (1) or (3).

The question of collineations (path-preserving continuous groups of transformations) in path-spaces for which the a^i *possess a formal expansion by polynomials homogeneous in* $\dot{x}$, *can be dealt with by methods similar to those used for manifolds with asymmetric affine connection. The particular connection, moreover, is represented by the coefficients of the quadratic terms in the expansion.*

References.

(1) D. D. Kosambi. Math Zeitschrift Bd. 37 (1933), pp. 608, 618.
(2) E. Cartan. Ibid, pp. 619, 622.
(3) L. P. Eisenhart. Non-Riemannian Geometry (1927), pp. 126, 132.

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